MATRIX THEORY AND IT'S APPLICATIONS

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ABSTRACT

The purpose of this paper is to show the details of matrix algebra, matrix factorization, definitions of upper and lower triangular matrices with several methods such as Doolittle method, Crout method, Choles method and examples of their application. We are aware of how important knowledge of matrix theory is, especially for those who take mathematics in whatever direction they study. Hence the greatest urge to write this paper with the hope that it will be useful to all those who need it..

Keywords. matrix theory, matrix factorization, Doolittle method, Crout method, Choles method

1.0 INTRODUCTION

Definition 1.1. Let A be an nxm matrix and λ a real number. The scalar product of λ and A is denoted by λA and is an nxm matrix with components for each $i = 1,2..n$ and each $j = 1,2...m$.

Theorem 1.2. Let A, B and C be nxm λ and μ real numbers. The following specifications are valid.

a) $A + B = B + A$
b) $(A + B) + C = A + (B + C)$
c) $A + 0 = 0 + A = A$
d) $A + (-A) = (-A) + A = 0$
e) $\lambda(A + B) = \lambda A + \lambda B$
f) $(\lambda + \mu)A = \lambda A + \mu A$
g) $\lambda(\mu A) = (\lambda\mu)A$
h) $1 \cdot A = A$

Definition 1.3. In the form of a matrix,

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = [a_{ij}]$$

is a rectangular arrangement of numbers.

The i-th row of A

$$[a_{i1} \quad a_{i2} \quad \cdots \quad a_{in}], 1 \leq i \leq n$$

and the j-th column of A

$$\begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix} \quad 1 \leq j \leq n$$

Definition 1.4. A matrix with m rows and n columns is called *an mxn matrix* or (m, n) $-$matrix for short.

Example 1.5. The following are matrices

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -2 & 4 \\ 0 & -3 & 5 \end{bmatrix}, \quad B = [1 \quad 5 \quad -2], \quad C = \begin{bmatrix} 3 \\ -2 \\ 1 \\ 5 \end{bmatrix}, \text{ and } \quad D = \begin{bmatrix} 0 & 2 \\ 2 & -1 \end{bmatrix}$$

Here, they are matrices of type A;$3x3$, B;$1x3$, C; $4x1$ and D;$2x2$.

Definition 1.6. If the number of rows of the matrix is equal to the number of columns, i.e. if $m = n$, we have a square *matrix of order n* or shorter (n) matrix.

Example 1.7.

$$A = \begin{bmatrix} 1 & 2 & -2 \\ 2 & 5 & 4 \\ 0 & -2 & 6 \end{bmatrix} \text{ and } B = \begin{bmatrix} x & 2 & -2 \\ 2 & z & 4 \\ 0 & -2 & y \end{bmatrix}$$

for the matrices to be equal, it is necessary and sufficient to be $x = 1, y = 6$ and $z = 5$.

Definition 1.8. Quadrant matrices are distinguished from diagonal matrices in which all elements of different indices $(i \neq j)$ are equal to zero.

$$\begin{bmatrix} a_{11} & 0 & 0 & \cdot & \cdot & 0 \\ 0 & a_{22} & 0 & \cdot & \cdot & 0 \\ 0 & 0 & a_{33} & \cdot & \cdot & 0 \\ \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot \\ 0 & 0 & 0 & \cdot & \cdot & a_{nn} \end{bmatrix}$$

It is said that the elements $a_{11}, a_{22}, a_{33} \cdots a_{nn}$ lie on the main diagonal of the matrix.

Diagonal matrices in which all elements on the main diagonal are equal to each other $(a_{11} = a_{22} = a_{33} = \cdots = a_{nn})$ are called *scalar matrices*, in particular, if $a = 1$, the scalar matrix is called *the unit matrix* and is denoted by the letter E.

$$E = \begin{bmatrix} 1 & 0 & 0 & \cdot & \cdot & 0 \\ 0 & 1 & 0 & \cdot & \cdot & 0 \\ 0 & 0 & 1 & \cdot & \cdot & 0 \\ \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot \\ 0 & 0 & 0 & \cdot & \cdot & 1 \end{bmatrix}$$

Definition 1.9. A matrix with all elements equal to zero is called a *zero matrix* and is denoted by 0.

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

Definition 1.10. Two matrices A and B are equal, i.e. $A = B$, if they have the same elements in the same position, i.e. they have the same number of rows and the same number of columns with identical elements.

Example 1.11.

$$\begin{bmatrix} 2 & 1 \\ 6 & 9 \\ 4 & 7 \end{bmatrix} = \begin{bmatrix} 4-2 & {}^{5}/_{5} \\ 2+4 & {}^{27}/_{3} \\ {}^{1}/_{3} \cdot 12 & (7) \cdot (1) \end{bmatrix}$$

2.0 OPERATIONS WITH MATRICES

Definition 2.1. (Addition of matrices): Two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ can be added, if they have the same number of rows and the same number of columns. In that case, the sum matrix is $C = A + B$, i.e. $c_{ij} = a_{ij} + b_{ij}, i = 1,2, \dots, m, j = 1,2, \dots, n,$ it is calculated so that the corresponding elements of those matrices are added separately.

Example 2.2.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} + \begin{bmatrix} 2 & 5 & -3 \\ 3 & -2 & 4 \\ 4 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1+2 & 2+5 & 3+(-3) \\ 4+3 & 5+(-2) & 6+4 \\ 7+4 & 8+1 & 9+2 \end{bmatrix} = \begin{bmatrix} 3 & 7 & 0 \\ 7 & 3 & 10 \\ 11 & 9 & 11 \end{bmatrix}$$

The rule of adding two matrices is valid for any finite number of addends.

Example 2.3.

$$\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} + \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2+3+1 \\ 3+1+1 \\ 1+1+1 \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \\ 3 \end{bmatrix}$$

The laws of commutation and association apply to the addition of matrices:

$$A + B = B + A$$

$$(A + B) + C = A + (B + C)$$

Definition 2.4. (**Subtraction of matrices)** The difference of two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ of the same type (mxn) is determined by subtracting the elements of the subtrahend from the corresponding elements of the minuemd.

Example 2.5. Determine the difference of matrices

$$A = \begin{bmatrix} 5 & 1 & 0 \\ 1 & 0 & 6 \\ 3 & 2 & 8 \end{bmatrix} \quad \text{i} \quad \mathrm{B} = \begin{bmatrix} 3 & 6 & 4 \\ 2 & 5 & 3 \\ 1 & 0 & 2 \end{bmatrix}$$

$$C = A - B = \begin{bmatrix} 5-3 & 1-6 & 0-4 \\ 1-2 & 0-5 & 6-3 \\ 3-1 & 2-0 & 8-2 \end{bmatrix} = \begin{bmatrix} 2 & -5 & -4 \\ -1 & -5 & 3 \\ 2 & 2 & 6 \end{bmatrix}$$

Definition 2.6. (**The product of a matrix and a scalar**): The product A α gives a matrix C, the elements of which are the products of the elements of the matrix A and the scalar α.

Example 2.7.

$$4 \cdot \begin{bmatrix} 4 & 6 \\ 3 & 2 \\ 5 & 1 \end{bmatrix} = \begin{bmatrix} 4 \cdot 4 & 4 \cdot 6 \\ 4 \cdot 3 & 4 \cdot 2 \\ 4 \cdot 5 & 4 \cdot 1 \end{bmatrix} = \begin{bmatrix} 16 & 24 \\ 12 & 8 \\ 20 & 4 \end{bmatrix}$$

Definition 2.8. (**Multiplying a matrix by a matrix**): The product AB of matrices A and B is defined only for the case when the number of columns of matrix A is equal to the number of rows of matrix B, so in the product AB the number of rows is always equal to the number of rows in the first factor A, while the number of columns is equal to the number of columns in the second factor B.

If the matrix A is a row-matrix

$$A = (a_1 a_2 \dots a_n)$$

and matrix B matrix-column

$$B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

then it is

$$C = A \cdot B = (a_1 a_2 \dots a_n) \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = [a_1 b_1 + a_2 b_2 + \cdots + a_b b_n]$$

Example 2.9.

$$(1 \quad 2 \quad 4) \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix} = [1 \cdot 2 + 2 \cdot 3 + 4 \cdot 5] = [28]$$

The number *a* can be multiplied by any matrix

$$\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} (a_1 a_2 a_3) = \begin{bmatrix} b_1 a_1 & b_1 a_2 & b_1 a_3 \\ b_2 a_1 & b_2 a_2 & b_2 a_3 \\ b_3 a_1 & b_3 a_2 & b_3 a_3 \end{bmatrix}$$

In the general case, we proceed as follows:

$$AB = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} \end{bmatrix}$$

The procedure follows by multiplying all the elements of the first row of matrix A with the corresponding elements of the first column of matrix B and then adding these products together to obtain the first term of the product. Proceeding in the same way with the same row of matrix A and the longest column of matrix B, we get the second term of the product required by the first row. We proceed in the same way with the elements of the second row of the matrix A and the volume of the columns oft the matrix B, so we get the first and second terms of the second row of the product AB.

The properties of matrix product are summarized in the following theorem.

Theorem 2.10.

a) $\mathrm{A(BC) = (AB)C}$
b) $\mathrm{ImB = B}$ and $\mathrm{BIn = B}$
c) $\mathrm{A(B + D) = AB + AD}$
d) $\mu\mathrm{(AB)} = (\mu\mathrm{A})\mathrm{B} = \mathrm{A}(\mu\mathrm{B})$

Definition 2.11.

If every $j = 2,3, \ldots, n$ and every $i = 1,2, \ldots . j - 1$ and $a_{ij} = 0$ in an nxn matrix A, the matrix A is called lower triangular matrix.

In an nxn matrix A, if every $j = 1,2, \ldots, n - 1$ and every $i = j + 1 \ldots$ and $a_{ij} = 0$, then the matrix A is called upper triangular matrix.

Definition 2.12. If we fold matrix A around its main diagonal, its columns will become rows, and its rows will become columns. This new matrix, which is called the transposed matrix with respect to the matrix A, is denoted by $A^T = \tilde{A}$. In this way (m, n) matrix A is transformed into (n, m) matrix $A^T = \tilde{A}$.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad A^T = \tilde{A} = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & \cdots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix}$$

Theorem 2.13

a) If we perform the transpose operation twice on the matrix A, the matrix A remains unchanged:

$$(A^T)^T = A$$

b) The transposed matrix of the sum of two matrices is equal to the sum of the transposed matrices:

$$(A+B)^T = A^T + B^T$$

c) The determinant of the matrix A is equal to the determinant of the matrix A^T:

$$detA = A = \det A^T = A^T.$$

d) The transposed matrix of the product of two matrices is equal to the product of transposed matrices taken in reverse order:

$$(AB)^T = B^T A^T$$

Example 2.14.

$$A = \begin{bmatrix} 2 & 6 & 4 \\ 3 & 5 & 1 \\ 4 & 2 & 3 \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} 2 & 3 & 4 \\ 6 & 5 & 2 \\ 4 & 1 & 3 \end{bmatrix}$$

Theorem 2.15. Let A and B be nxn matrices

a) If all components in any row or column of A nxn are zero then it is $detA = 0$
b) If the A^T matrix is obtained from the matrix A $(E_i) \Leftrightarrow)(E_j)$ with $j \neq i$, then

$$detA^T = detA$$

c) If two of A are the same, detA=0.
d) If the A^T matrix is obtained from the A matrix $(\lambda E_I) \rightarrow (E_i)$ by the operation

$$\det(A^T) = \lambda detA$$

e) $\det(AB) = \det(A)\det(B)$

Theorem 2.16. If the matrix A is a lower triangular or an upper triangular matrix

$$\det(A) = \prod_{i=1}^{n} a_{ii}$$

we can write the linear derlin system as $Ax = b$

Example 2.17.

$$E_1: x_1 - 2x_2 + 2x_3 = 3$$

$$E_2: \ 2x_1 + x_2 + x_3 = 0$$

$$E_3: \ x_1 + 0 + x_3 = -2$$

$$\begin{bmatrix} 1 & -2 & 2 \\ 2 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix}$$

$$(E_1 + 2E_2) \rightarrow (E_1) \qquad \begin{bmatrix} 5 & 0 & 4 \\ 2 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix}$$

$$(E_1 - 5E_3) \rightarrow (E_1) \quad \begin{bmatrix} 0 & 0 & -1 \\ 2 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 13 \\ 0 \\ -2 \end{bmatrix}$$

$$(E_2 - 2E_3) \rightarrow (E_2) \quad \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 13 \\ 4 \\ -2 \end{bmatrix}$$

$$(E_1 + E_3) \rightarrow (E_3) \quad \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & -1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 13 \\ 4 \\ 11 \end{bmatrix}$$

$$(E_1 - E_2) \rightarrow (E_2) \quad \begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 13 \\ 9 \\ 11 \end{bmatrix}$$

$x_3 = -13$, $x_2 = -9$, $x_1 = 11$

Another way we can follow is to add the B vector to the right of the A matrix to obtain the $\tilde{A}$ $n\text{x}(n+1)$ matrix.

$$\bar{A} = [A, b] = \bar{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & \vdots & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & \vdots & b_2 \\ \vdots & \vdots & \dots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & \vdots & b_n \end{bmatrix}$$

Let's solve the same ogre in this way.

$$\bar{A} = \begin{bmatrix} 1 & -2 & 2 & \vdots & 3 \\ 2 & 1 & 1 & \vdots & 0 \\ 1 & 0 & 1 & \vdots & -2 \end{bmatrix}$$

$$(E_1 + 2E_2) \rightarrow (E_1) \quad \begin{bmatrix} 5 & 0 & 4 & \vdots & 3 \\ 2 & 1 & 1 & \vdots & 0 \\ 1 & 0 & 1 & \vdots & -2 \end{bmatrix}$$

$$(E_1 - 5E_3) \rightarrow (E_1) \quad \begin{bmatrix} 0 & 0 & -1 & \vdots & 13 \\ 2 & 1 & 1 & \vdots & 0 \\ 1 & 0 & 1 & \vdots & -2 \end{bmatrix}$$

$$(E_2 - 2E_3) \rightarrow (E_2) \quad \begin{bmatrix} 0 & 0 & -1 & \vdots & 13 \\ 2 & 1 & -1 & \vdots & 0 \\ 1 & 0 & 0 & \vdots & -2 \end{bmatrix}$$

$$(E_1 + E_3) \rightarrow (E_3) \quad \begin{bmatrix} 0 & 0 & -1 & \vdots & 13 \\ 0 & 1 & -1 & \vdots & 4 \\ 1 & 0 & 0 & \vdots & 11 \end{bmatrix}$$

$$(E_1 - E_2) \rightarrow (E_2) \quad \begin{bmatrix} 0 & 0 & -1 & \vdots & 13 \\ 0 & -1 & 0 & \vdots & 9 \\ 1 & 0 & 0 & \vdots & 11 \end{bmatrix}$$

$x_3 = -13$, $x_2 = -9$, $x_1 = 11$

Definition 2.18. A square matrix is called symmetric if its elements, which lie symmetrically with respect to the main diagonal, are mutually equal.

A matrix is cosimetric, if its elements are symmetrically positioned with respect to the main diagonal, equal in size and opposite in sign.

Symmetric matrix $A = \breve{A}$ and cosimetric matrix $A = -\breve{A}$

Definition 2.19. A square matrix A^{-1} is called inverse with respect to a square matrix A, if

$$AA^{-1} = A^{-1} = I$$

where I is the unit matrix.

Definition 2.20. If an A matrix does not have an inverse, the A matrix is called a singular matrix.

Theorem 2.21. Let A and B be inverse matrices. The following specifications are valid.

a) $(A^{-1})^{-1} = A$
b) $(AB)^{-1} = B^{-1}A^{-1}$
c) $(A^T)^{-1} = (A^{-1})^T$
d) $Ax = b \rightarrow x = A^{-1}b$

The procedure for calculating the inverse matrices:

1) We calculate for the given matrix A the value of the determinant, i.e.
$$\Delta = \det A = A$$
2) We transpose the given matrix A to obtain the matrix $\breve{A}$.
3) For each element of the matrix $\breve{A}$, we calculate, row by row, the corresponding cofactors or algebraic complements, i.e. subdeterminants, which we obtain by crossing out the column and the row in which the element in question lies, and we take the plus and minus signs as cofactors alternately, regardless of the sign of the element for which we calculate the cofactor.
4) In the matrix A, we replace each element with the corresponding cofactor.
5) If we divide each member of the thus obtained matrix by $\Delta = \det A$, we will get the required matrix A^{-1} inverse with respect to the given matrix A.
6) Let's do an experiment: it must be

$$AA^{-1} = A^{-1} = I = unit\ matrix\ = 1$$

Example 2.22.

A square matrix is given

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 5 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

Determine the inverse matrix A^{-1}

Solution:

1) We count $\Delta = \det A = A$

$$\Delta = \begin{bmatrix} 1 & 2 & 3 \\ 5 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 3 \\ 5 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 3 \\ 5 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

$$\Delta = 1(5 \cdot 9 - 6 \cdot 8) - 2(5 \cdot 9 - 6 \cdot 7) + 3(5 \cdot 8 - 7 \cdot 5)$$

$$\Delta = 1(45 - 48) - 2(45 - 42) + 3(40 - 35)$$

$$\Delta = -3 - 6 + 15 = 6 \neq 0$$

The given matrix A is regular, so it has an inverse matrix A^{-1}

2) We transpose the matrix A:

$$A^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 5 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$$

3) We calculate the cofactors for matrix $\check{A}$.

$$A_{11} = \begin{vmatrix} a_{22} & a_{32} \\ a_{23} & a_{33} \end{vmatrix} = +\begin{vmatrix} 5 & 8 \\ 6 & 9 \end{vmatrix} = +(5 \cdot 9 - 6 \cdot 8) = 45 - 48 = -3$$

$$A_{21} = -\begin{vmatrix} a_{12} & a_{32} \\ a_{13} & a_{33} \end{vmatrix} = -\begin{vmatrix} 2 & 8 \\ 3 & 9 \end{vmatrix} = -(2 \cdot 9 - 3 \cdot 8) = -(18 - 24) = -(-6) = 6$$

$$A_{31} = \begin{vmatrix} a_{12} & a_{22} \\ a_{13} & a_{23} \end{vmatrix} = \begin{vmatrix} 2 & 5 \\ 3 & 6 \end{vmatrix} = (2 \cdot 6 - 3 \cdot 5) = (12 - 15) = -3$$

$$A_{12} = -\begin{vmatrix} a_{21} & a_{31} \\ a_{23} & a_{33} \end{vmatrix} = -\begin{vmatrix} 5 & 7 \\ 6 & 9 \end{vmatrix} = -(5 \cdot 9 - 6 \cdot 7) = -(45 - 42) = -3$$

$$A_{22} = \begin{vmatrix} a_{11} & a_{31} \\ a_{13} & a_{33} \end{vmatrix} = \begin{vmatrix} 1 & 7 \\ 3 & 9 \end{vmatrix} = (1 \cdot 9 - 3 \cdot 7) = 9 - 21 = -12$$

$$A_{32} = -\begin{vmatrix} a_{11} & a_{21} \\ a_{13} & a_{23} \end{vmatrix} = -\begin{vmatrix} 1 & 5 \\ 3 & 6 \end{vmatrix} = -(1 \cdot 6 - 3 \cdot 5) = -(6 - 15) = -(-9) = 9$$

$$A_{13} = \begin{vmatrix} a_{21} & a_{31} \\ a_{22} & a_{32} \end{vmatrix} = \begin{vmatrix} 5 & 7 \\ 5 & 8 \end{vmatrix} = (5 \cdot 8 - 5 \cdot 7) = (40 - 35) = 5$$

$$A_{23} = -\begin{vmatrix} a_{11} & a_{31} \\ a_{12} & a_{32} \end{vmatrix} = -\begin{vmatrix} 1 & 7 \\ 2 & 8 \end{vmatrix} = -(1 \cdot 8 - 2 \cdot 7) = -(8 - 14) = -(-6) = 6$$

$$A_{33} = \begin{vmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{vmatrix} = \begin{vmatrix} 1 & 5 \\ 2 & 5 \end{vmatrix} = (1 \cdot 5 - 2 \cdot 5) = 5 - 10 = -5$$

4) We get it

$$\begin{bmatrix} -3 & 6 & -3 \\ -3 & -12 & 9 \\ 5 & 6 & -5 \end{bmatrix}$$

5) We divide each element by $\Delta = detA = A = 6.$

$$A^T = \begin{bmatrix} \frac{-3}{6} & \frac{6}{6} & \frac{-3}{6} \\ \frac{-3}{6} & \frac{-12}{6} & \frac{9}{6} \\ \frac{5}{6} & \frac{6}{6} & \frac{-5}{6} \end{bmatrix}$$

We get:

$$A^T = \begin{bmatrix} \frac{-1}{2} & 1 & \frac{-1}{2} \\ \frac{-1}{2} & -2 & \frac{3}{2} \\ \frac{5}{6} & 1 & \frac{-5}{6} \end{bmatrix}$$

6) Let's do an experiment:

$$AA^T = \begin{bmatrix} 1 & 2 & 3 \\ 5 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} \frac{-1}{2} & 1 & \frac{-1}{2} \\ \frac{-1}{2} & -2 & \frac{3}{2} \\ \frac{5}{6} & 1 & \frac{-5}{6} \end{bmatrix} = The\ law\ of\ matrix\ multiplication$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = R = 1$$

3.0 FORWARD SUBSTITATION

$$\begin{aligned} a_{11}x_1 \qquad\qquad\qquad &= b_1 \\ a_{21}x_1 + a_{22}x_2 \qquad\quad &= b_2 \\ \vdots \qquad \vdots \qquad\qquad &\ \ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{aligned}$$

1) $a_{11}x_1 = b_1 \Longrightarrow x_1 = \frac{b_1}{a_1}$
2) $a_{21}x_1 + a_{22}x_2 = b_2 \Longrightarrow x_2 = \frac{(b_2 - a_{21}x_1)}{a_{22}}$

$\vdots$

$$\sum_{j=1}^{i} a_{ij}x_j = b_i \Longrightarrow b_i = \sum_{j=1}^{i} a_{ij}x_j + a_{ii}x_i \Longrightarrow x_i = \frac{\left(b_i - \sum_{j=1}^{i-1} a_{ij}x_j\right)}{a_{ii}}$$

Repeating the following from $i \ = \ 2$ to n

$$x_i = \frac{\left(b_i - \sum_{j=1}^{i-1} a_{ij}x_j\right)}{a_{ii}}$$

Provided that $a_{ii} \neq 0$ for every $i = 1,2,\dots n$.

3.2. Reverse substitution

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$

$$a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$

$$\vdots$$

$$a_{nn}x_n = b_n$$

$$a_{nn}x_n = b_n \Longrightarrow x_n = \frac{b_n}{a_{nn}}$$

$$a_{n-1,n-1}x_{n-1} + a_{n-1,n}x_n = b_{n-1} \Longrightarrow x_{n-1} = \frac{b_{n-1} - a_{n-1,n}x_n}{a_{n-1,n-1}}$$

$$\sum_{j=1}^{n} a_{ij}x_j = b_i \Rightarrow \sum_{j=i+1}^{n} a_{ij}x_j + a_{ii}x_{ii} = b_i \Rightarrow x_i = \frac{\left(b_i - \sum_{j=i+1}^{n} a_{ij}x_j\right)}{a_{ii}}$$

$$x_n = \frac{b_n}{a_{nn}}$$

Let the following repeat from $i = (n-1)$ to 1, with steps of -1 each.

$$x_i = \frac{(b_i - \sum_{j=i+1}^{n} a_{ij}x_j)}{a_{ii}}$$

Example 3.2.1.

$$E_1: 2,00x_1 + 2,00x_2 + 3,00x_3 + 3,00x_4 = 1,00$$

$$E_2: 2,00x_1 + 3,00x_2 + 3,00x_3 + 2,00x_4 = 1,00$$

$$E_3: 5,00x_1 + 3,00x_2 + 7,00x_3 + 9,00x_4 = 1,00$$

$$E_4: 3,00x_1 + 2,00x_2 + 4,00x_3 + 7,00x_4 = 1,00$$

$$(E_2 - 1,00E_1) \rightarrow (E_2)\,, \qquad (E_3 - 2,50E_1) \rightarrow (E_3)\,, (E_4 - 1,50E_1) \rightarrow (E_4)\,,$$

$$\begin{aligned} E_1&: 2{,}00x_1 + 2{,}00x_2 + 3{,}00x_3 + 3{,}00x_4 = 1{,}00 \\ E_2&: \quad 1{,}00x_2 \quad - 1{,}00x_4 = 0{,}00 \\ E_3&: \quad -2{,}00x_2 - 0{,}50x_3 + 1{,}50x_4 = -1{,}50 \\ E_4&: \quad -1{,}00x_2 - 0{,}50x_3 + 2{,}50x_4 = -0{,}50 \end{aligned}$$

$$(E_3 + 2{,}00E_2) \to (E_3)\,, \qquad (E_4 + 1{,}00E_2) \to (E_4)\,,$$

$$\begin{aligned} E_1&: 2{,}00x_1 + 2{,}00x_2 + 3{,}00x_3 + 3{,}00x_4 = 1{,}00 \\ E_2&: \quad 1{,}00x_2 \quad - 1{,}00x_4 = 0{,}00 \\ E_3&: \quad -0{,}50x_3 - 0{,}50x_4 = -1{,}50 \\ E_4&: \quad -0{,}50x_3 + 1{,}50x_4 = -0{,}50 \end{aligned}$$

$$(E_4 - 1{,}00E_3) \to (E_4)$$

$$\begin{aligned} E_1&: 2{,}00x_1 + 2{,}00x_2 + 3{,}00x_3 + 3{,}00x_4 = 1{,}00 \\ E_2&: \quad 1{,}00x_2 \quad - 1{,}00x_4 = 0{,}00 \\ E_3&: \quad -0{,}50x_3 - 0{,}50x_4 = -1{,}50 \\ E_4&: \quad 2{,}00x_4 = 1{,}00 \end{aligned}$$

the rest can be solved by reverse substitution.

$$\begin{aligned} E_1&: 2{,}00x_1 + 2{,}00x_2 + 3{,}00x_3 + 3{,}00x_4 = 1{,}00 \\ E_2&: \quad 1{,}00x_2 \quad - 1{,}00x_4 = 0{,}00 \\ E_3&: \quad -0{,}50x_3 - 0{,}50x_4 = -1{,}50 \\ E_4&: \quad x_4 = \frac{1{,}00}{2{,}00} \end{aligned}$$

$$\begin{aligned} E_1&: 2{,}00x_1 + 2{,}00x_2 + 3{,}00x_3 + 3{,}00x_4 = 1{,}00 \\ E_2&: \quad 1{,}00x_2 \quad - 1{,}00 \cdot \tfrac{1{,}00}{2{,}00} = 0{,}00 \\ E_3&: \quad -0{,}50x_3 - 0{,}50 \cdot \frac{1{,}00}{2{,}00} = -1{,}50 \\ E_4&: \quad x_4 = \frac{1{,}00}{2{,}00} \end{aligned}$$

$$E_1: 2{,}00x_1 + 2{,}00x_2 + 3{,}00x_3 + 3{,}00x_4 = 1{,}00$$
$$E_2: \quad 1{,}00x_2 \quad - 0{,}50 = 0{,}00$$
$$E_3: \quad - 0{,}50x_3 - 0{,}25 = -1{,}50$$
$$E_4: \quad x_4 = \frac{1{,}00}{2{,}00}$$

$$E_1: 2{,}00x_1 + 2{,}00x_2 + 3{,}00x_3 + 3{,}00x_4 = 1{,}00$$
$$E_2: \quad 1{,}00x_2 \quad = 0{,}50$$
$$E_3: \quad - 0{,}50x_3 = -1{,}25$$
$$E_4: \quad x_4 = 0{,}50$$

$$E_1: 2{,}00x_1 + 2{,}00 \cdot 0{,}50 + 3{,}00 \cdot 2{,}50 + 3{,}00 \cdot 0{,}50 = 1{,}00$$
$$E_2: \quad x_2 = 0{,}50$$
$$E_3: \quad x_3 = 2{,}50$$
$$E_4: \quad x_4 = 0{,}50$$

$$E_1: 2{,}00x_1 + 1{,}00 + 7{,}50 + 1{,}50 \quad = 1{,}00$$
$$E_2: \quad x_2 = 0{,}50$$
$$E_3: \quad x_3 = 2{,}50$$
$$E_4: \quad x_4 = 0{,}50$$

$$E_1: \quad 2{,}00x_1 \quad = 1{,}00 - 10{,}00$$
$$E_2: \quad x_2 = 0{,}50$$
$$E_3: \quad x_3 = 2{,}50$$
$$E_4: \quad x_4 = 0{,}50$$

$$E_1: \quad x_1 = -4{,}50$$
$$E_2: \quad x_2 = 0{,}50$$
$$E_3: \quad x_3 = 2{,}50$$
$$E_4: \quad x_4 = 0{,}50$$

Another way to solve a linear equivalent system is the *Gaussian elimination method* of inverse substitution, which we explained earlier.

3.3. Gauss-Jordan Method

The Gauss-Jordan method, which is a slightly different form of the Gaussian method, is a frequently used method that is worth focusing on. In this method, the components above the diagonal are also zeroed in between creating zeros below the diagonal. Therefore, there is no need for reverse substitution, and the solution is obtained by dividing the vector remaining on the right side after the elimination operations into the corresponding diagonal components on the left. If it is formulated more mathematically, as in the equivalent Gauss elimination method, it is used to eliminate xii from the equivalents $E_{i+1}, E_{i+2}, \dots \dots, E_n$, as well as eliminating xi from the equivalents $E_1, E_2, \dots \dots E_{i-1}$. It is used to do.

After the reduction process, the generalized matrix $[A, b]$ is reduced to.

$$[A,b] = \begin{bmatrix} a_{1,1}^{(1)} \dots & \dots \dots \dots \dots \dots \dots \dots \dots a_{1,n+1}^{(n+1)} \\ & a_{2,2}^{(2)} \dots \dots \dots \dots \dots \dots \dots \dots a_{2,n+1}^{(n+1)} \\ & \dots \dots \dots \dots \dots \dots \dots \dots \\ & a_{n-1,n-1}^{(n-1)} \dots \dots \dots a_{n-1,n+1}^{(n+1)} \\ & a_{n,n}^{(n)} \dots \dots \dots a_{n,n+1}^{(n)} \end{bmatrix}$$

Most of the time, when using the Gauss-Jordan method, the diagonal components are also set to 1 during the reduction process. This makes it a coefficient matrix. When this is done, it means that the right-side vector has been reduced to the solution vector. All the "pivoting" process that will be explained later for Gauss can also be done for Gauss-Jordan.

Example 3.3.1.

$$E_1: 2{,}00x_1 + 2{,}00x_2 + 3{,}00x_3 + 3{,}00x_4 = 1{,}00$$

$$E_2: 2{,}00x_1 + 3{,}00x_2 + 3{,}00x_3 + 2{,}00x_4 = 1{,}00$$

$$E_3: 5{,}00x_1 + 3{,}00x_2 + 7{,}00x_3 + 9{,}00x_4 = 1{,}00$$

$$E_4: 3{,}00x_1 + 2{,}00x_2 + 4{,}00x_3 + 7{,}00x_4 = 1{,}00$$

$$\begin{bmatrix} 2{,}00 & 2{,}00 & 3{,}00 & 3{,}00 & \vdots & 1{,}00 \\ 2{,}00 & 3{,}00 & 3{,}00 & 2{,}00 & \vdots & 1{,}00 \\ 5{,}00 & 3{,}00 & 7{,}00 & 9{,}00 & \vdots & 1{,}00 \\ 3{,}00 & 2{,}00 & 4{,}00 & 7{,}00 & \vdots & 1{,}00 \end{bmatrix} \quad (0{,}50E_1) \to (E_1)$$

$$\begin{bmatrix} 1{,}00 & 1{,}00 & 1{,}50 & 1{,}50 & \vdots & 0{,}50 \\ 2{,}00 & 3{,}00 & 3{,}00 & 2{,}00 & \vdots & 1{,}00 \\ 5{,}00 & 3{,}00 & 7{,}00 & 9{,}00 & \vdots & 1{,}00 \\ 3{,}00 & 2{,}00 & 4{,}00 & 7{,}00 & \vdots & 1{,}00 \end{bmatrix} \quad \begin{matrix} (E_2 - 2{,}00E_1) \to (E_2) \\ (E_3 - 5{,}00E_1) \to (E_3) \\ (E_4 - 3{,}00E_1) \to (E_4) \end{matrix}$$

$$\begin{bmatrix} 1{,}00 & 1{,}00 & 1{,}50 & 1{,}50 & \vdots & 0{,}50 \\ 0{,}00 & 1{,}00 & 0{,}00 & -1{,}00 & \vdots & 0{,}00 \\ 0{,}00 & -2{,}00 & -0{,}50 & 1{,}50 & \vdots & -1{,}50 \\ 0{,}00 & -1{,}00 & -0{,}50 & 2{,}50 & \vdots & -0{,}50 \end{bmatrix} \quad \begin{matrix} (E_3 + 2{,}00E_2) \to (E_3) \\ (E_4 + 1{,}00E_2) \to (E_4) \\ (E_1 - 1{,}00E_2) \to (E_1) \end{matrix}$$

$$\begin{bmatrix} 1,00 & 1,00 & 1,50 & 1,50 & \vdots & 0,50 \\ 0,00 & 1,00 & 0,00 & -1,00 & \vdots & 0,00 \\ 0,00 & 0,00 & -0,50 & -0,50 & \vdots & -1,50 \\ 0,00 & 0,00 & -0,50 & -1,50 & \vdots & -0,50 \end{bmatrix} (-2,00E_3) \to (E_3\ )$$

$$\begin{bmatrix} 1,00 & 1,00 & 1,50 & 1,50 & \vdots & 0,50 \\ 0,00 & 1,00 & 0,00 & -1,00 & \vdots & 0,00 \\ 0,00 & 0,00 & 1,00 & 1,00 & \vdots & 3,00 \\ 0,00 & 0,00 & -0,50 & 1,50 & \vdots & -0,50 \end{bmatrix} \begin{matrix} (E_4 + 0,50E_2) \to (E_4) \\ (E_1 - 1,50E_3) \to (E_1) \\ (E_2 - 0,00E_3) \to (E_2) \end{matrix}$$

$$\begin{bmatrix} 1,00 & 1,00 & 1,50 & 1,50 & \vdots & 0,50 \\ 0,00 & 1,00 & 0,00 & -1,00 & \vdots & 0,00 \\ 0,00 & 0,00 & 1,00 & 1,00 & \vdots & 3,00 \\ 0,00 & 0,00 & 0,00 & 2,00 & \vdots & 1,00 \end{bmatrix} (0,50E_4) \to (E_4)$$

$$\begin{bmatrix} 1,00 & 1,00 & 1,50 & 1,50 & \vdots & 0,50 \\ 0,00 & 1,00 & 0,00 & -1,00 & \vdots & 0,00 \\ 0,00 & 0,00 & 1,00 & 1,00 & \vdots & 3,00 \\ 0,00 & 0,00 & 0,00 & 1,00 & \vdots & 0,50 \end{bmatrix} \begin{matrix} (E_3 - 1,00E_4) \to (E_3) \\ (E_2 + 1,00E_1) \to (E_2) \\ (E_1 - 1,00E_3) \to (E_1) \end{matrix}$$

Comparison of Gauss and Gauss-Jordan methods in terms of elementary operations gives the following table. In the table, n is the number of unknowns.

Method	Addition-Subtraction	Multiplication	Sharing
Gauss	$n(n-1)(2n+5)/6$	$n(n-1)(2n+5)/6$	$n(n+1)/2$
Gauss-Jordan	$n(n-1)(n+1)/2$	$n(n-1)(n+1)/2$	$n(n+1)/2$

4.0 LEADING PRINCIPAL SUBMATRICES

Definition 4.1. Substituting the matrix A_i into its nxn inverse, where $i = 1,2,\ldots,n$, the necessary and sufficient conditions for the Gauss elimination method to be applied without the necessity of row swapping are that all the first fundamental submatrices of the matrix A are non-singular.

$$A_1 = [a_{11}] = A_1^{(1)} \Rightarrow detA_1 = detA_1^{(1)}$$

$$A_2 = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$detA_2 = detA_2^{(2)}$$

$$det\left(\begin{bmatrix} 1 & 0 \\ -m_{21} & 1 \end{bmatrix}\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\right) = det\begin{bmatrix} 1 & 0 \\ -m_{21} & 1 \end{bmatrix} det\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$detA_2 = detA_2^{(2)} = a_{22}^{(2)} detA_1$$

$$A_3 = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \to A_3^{(3)} = \begin{bmatrix} a_{11}^{(1)} & a_{12}^{(1)} & a_{13}^{(1)} \\ 0 & a_{22}^{(2)} & a_{23}^{(2)} \\ 0 & 0 & a_{33}^{(3)} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -m_{32} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -m_{21} & 1 & 0 \\ -m_{31} & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = A_3^{(3)}$$

$$detA_3 = detA_3^{(3)} = a_{33}^{(3)} a_{22}^{(2)} a_{11}^{(1)} = a_3^{(3)} detA_2^{(2)} = a_{33}^{(3)} detA_2$$

$$detA_k = detA_k^{(k)} = a_{kk}^{(k)} detA_{k-1}^{(k-1)} = a_{kk}^{(k)} detA_{k-1}$$

Example 4.2.

$$\det A_1 = |2| = 2 \neq 0$$

$$\det A_2 = \begin{vmatrix} 2 & 2 \\ 2 & 3 \end{vmatrix} = 2 \neq 0$$

$$A = \begin{bmatrix} 2 & 2 & 3 & 3 \\ 2 & 3 & 3 & 2 \\ 5 & 3 & 7 & 9 \\ 3 & 2 & 4 & 7 \end{bmatrix} \to \quad \det A_3 = \begin{vmatrix} 2 & 2 & 3 \\ 2 & 3 & 3 \\ 5 & 3 & 7 \end{vmatrix} = -1 \neq 0$$

$$\det A_4 = \begin{bmatrix} 2 & 2 & 3 & 3 \\ 2 & 3 & 3 & 2 \\ 5 & 3 & 7 & 9 \\ 3 & 2 & 4 & 7 \end{bmatrix} = \det A = 6 \neq 0$$

As I have seen, all the first fundamental submatrices of this matrix are non-singular. Here, the necessary and sufficient conditions for the first basic sub-matrix to be non-singular are that the $a_{ii}^{(i)}$ for each $i = 1,2,3...,n$ are non-zero. if $a_{kk}^{(k)} = 0$ then $detA_k = 0$

Definition 4.3: Let A nxn matrix. If inequality $|a_{ii}| > \sum_{j=1}^{n} |a_{ij}|$occurs for every $i = 1,2,3...,n$, matrix A is called "strict diagonal dominant matrix".

Example 4.4:

$$A = \begin{bmatrix} 4 & -2 & -1 & 0 \\ 0 & -6 & 3 & -2 \\ 1 & 0 & 5 & -1 \\ 3 & -2 & -1 & 7 \end{bmatrix} \to \begin{array}{c} |a_{11}| = 4 > |-2| + |-1| + |0| = 3 \\ |a_{22}| = -6 > |0| + |3| + |-1| = 5 \\ |a_{33}| = 5 > |1| + |0| + |-1| = 2 \\ |a_{44}| = 7 > |3| + |-2| + |-1| = 6 \end{array}$$

A is the exact diagonal dominant matrix.

Theorem 4.5: If matrix A is a strictly diagonal dominant matrix of kxk, then A_{k-1} is a strictly diagonal dominant matrix.

Proof 4.5:

$\forall i = 1,2,3, ...,k$ for $|a_{ii}| > \sum_{\substack{j=1 \\ \neq i}}^{k} |a_{ij}|$ the feature is valid.

$$\forall i = 1,2,3, \dots, k-1 \text{ for } |a_{ii}| > \sum_{i=1}^{k} |a_{ij}|$$

$$\Rightarrow A_{k-1} \text{ the exact diagonal is the dominant matrix}$$

5.0 MATRIX FACTORIZATION

Theorem 5.1: Let's consider an nxn matrix A, all whose first principal submatrices are non-singular. Matrix A, where L is a lower triangular matrix and U is an upper triangular matrix, can be factored as $A = LU$, and there is only one way to do this.

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$AX = b$$

$$A = L_1 U_1 = L_2 U_2$$

$\det L_1 = \det L_2 = 1$ where is $\det A = \det U_1 = \det U_2$, A non-singular$\det U_1 = \det U_2 \neq 0$, so there is U_1^{-1} and U_2^{-1}.

In other words, U_1 and U_2 are also non-singular.

$$L_1 U_1 = L_2 U_2 \Rightarrow U_1 U_2^{-1} = L_1^{-1} L_2$$

$$U_1 U_2^{-1} = L_1^{-1} L_2 = I$$

$$U_1 = U_2, L_1 = L_2 \text{ are obtained.}$$

Then the factorization of A is unique, with U being an upper triangular matrix and L being a lower triangular matrix.

6.0 DOOLITTLE METHOD

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & \cdots & 0 \\ l_{21} & 1 & \ddots & \cdots & \vdots \\ l_{31} & l_{32} & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & 1 & 0 \\ l_{n1} & l_{n2} & \cdots & l_{n,n+1} & 1 \end{bmatrix} \begin{bmatrix} U_{11} & U_{12} & \cdots & \cdots & U_{1n} \\ 0 & U_{22} & \cdots & \cdots & U_{2n} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & U_{nn} \end{bmatrix}$$

By matrix-matrix multiplication

The first row of matrix L is completely known and has only one non-0 component. Let's multiply the first row of the L matrix by all the columns of the U matrix.

$$a_{1j} = U_{1j}\,, \quad j = 1,2,3 \dots, n$$

Now let's multiply all the rows of matrix L starting from the second by the first column of U.

$$a_{i1} = l_{i1} U_{11} \quad ; \quad i = 2,3, \dots, n$$

$$l_{i1} = \frac{a_{i1}}{U_{11}} \ ; \ i = 2,3, \dots., n$$

If we continue in a similar way, the second row of U and then the second column of L are determined. Now let's try to generalize what we did. Let $(k-1)$ rows of U and $(k-1)$ columns of L be determined.

$$a_{kj} = \sum_{m=1}^{k} l_{km} U_{mj} \qquad j = k, k+1, \dots., n$$

$$a_{kj} = \sum_{m=1}^{k-1} l_{km} U_{mj} + U_{kj} \qquad j = k, k+1, \dots, n$$

Note that in the $\sum$ operation, all l_{km} are in the first $(k-1)$ column of each L matrix.

At the same time, $\sum$ all U_{mj} in the process are in the first $(k-1)$ row of the U matrix. Therefore, their values are known. In other words, $\sum$ all values and current in the process are known.

$$U_{kj} = a_{kj} - \sum_{m=1}^{k-1} l_{km} U_{mj} \qquad j = k, k+1, \dots, n$$

The matrix U has k rows.

This time, if we multiply all the rows after k of the L matrix with the k columns of the U matrix, we get the following equation.

$$a_{ik} = \sum_{m=1}^{k} l_{im} U_{mk} \ , \quad i = k+1, \dots, n$$

$$a_{ik} = \sum_{m=1}^{k-1} l_{im} U_{mk} + l_{ik} U_{kk} \ , \qquad i = k+1, \dots., n$$

$\sum$ all l_{im}'s in the process are in the first $(k-1)$ column of the L matrix and therefore their values are known. $\sum$ all U_{mk}'s in the process are in the first $(k-1)$ row of the U matrix and therefore their values are known.

$$l_{ik} U_{kk} = a_{ik} - \sum_{m=1}^{k-1} l_{im} U_{mk} \ , \quad i = k+1, \dots, n$$

$$\Rightarrow l_{ik} = (a_{ik} - \sum_{m=1}^{k-1} l_{im} U_{mk}) / U_{kk} \ , \quad i = k+1, \dots, n$$

Example 6.1.

$$A=\begin{bmatrix}6&2&1&-1\\2&4&1&0\\1&1&4&-1\\-1&0&-1&3\end{bmatrix}=\begin{bmatrix}1&0&0&0\\l_{21}&1&0&0\\l_{31}&l_{32}&1&0\\l_{41}&l_{42}&l_{43}&1\end{bmatrix}\begin{bmatrix}U_{11}&U_{12}&U_{13}&U_{14}\\0&U_{22}&U_{23}&U_{24}\\0&0&U_{33}&U_{34}\\0&0&0&U_{44}\end{bmatrix}$$

$$U_{11}=6, U_{12}=2, U_{13}=1, U_{14}=-1$$

$$l_{21}U_{11}=2, l_{31}U_{11}=1, l_{41}U_{11}=-1$$

$$l_{21}=\frac{2}{U_{11}}, \quad l_{31}=\frac{1}{U_{11}}, l_{41}=\frac{-1}{U_{11}}$$

$$l_{21}=\frac{2}{6}=\frac{1}{3}, \quad l_{31}=\frac{1}{6}, l_{41}=\frac{-1}{6}$$

$$L=\begin{bmatrix}1&0&0&0\\\frac{1}{3}&1&0&0\\\frac{1}{6}&l_{32}&1&0\\-\frac{1}{6}&l_{42}&l_{43}&1\end{bmatrix}, \quad U=\begin{bmatrix}6&2&1&-1\\0&U_{22}&U_{23}&U_{24}\\0&0&U_{33}&U_{34}\\0&0&0&U_{44}\end{bmatrix}$$

Let's multiply the 2 rows of L by all the columns of U.

$$\frac{2}{3}+U_{22}=4\Rightarrow U_{22}=\frac{10}{3}$$

$$\frac{1}{3}+U_{23}=1\Rightarrow U_{23}=\frac{2}{3}$$

$$-\frac{1}{3}+U_{24}=0\Rightarrow U_{24}=\frac{1}{3}$$

We multiply all the rows of L after 2 by the 2 columns of U.

$$\frac{2}{6}+\frac{10}{3}l_{32}=1\Rightarrow l_{32}=\frac{1}{5}$$

$$\left(-\frac{2}{6}\right)+\frac{10}{3}l_{42}=0\Rightarrow l_{42}=\frac{1}{10}$$

If 3 rows of L are multiplied by the columns of U starting from 3;

$$\frac{1}{6}\cdot 1+\frac{1}{5}\cdot\frac{2}{3}+U_{33}=4\Rightarrow U_{33}=\frac{37}{10}$$

$$\frac{1}{6}\cdot(-1)+\frac{1}{5}\cdot\frac{1}{3}+U_{34}=-1\Rightarrow U_{33}=-\frac{9}{10}$$

To determine the last row of L, we multiply the last row of L by the 3rd and 4th columns of U.

$$l_{42} = -\frac{9}{2}, \quad U_{44} = \frac{191}{74}$$

Theorem 6.2. Let A be a matrix of 1 and all its first principal submatrices are non-singular, L is a lower triangular matrix, V is an upper triangular matrix, and D is a diagonal matrix. Since $A = LDV$, it can be factored and there is only one way to do this.

Proof: There is only one way to do this: L is lower triangular, U is upper triangular matrices, and A=LU is their multiplicative separation.

The diagonal elements of matrix D are the same as the diagonal elements of matrix U.

$$D = \begin{bmatrix} U_{11} & & 0 \\ & U_{22} & \\ 0 & \ddots & U_{nn} \end{bmatrix} \Rightarrow D^{-1} = \begin{bmatrix} {}^{1}/{U_{11}} & & 0 \\ & {}^{1}/{U_{22}} & \\ 0 & \ddots & {}^{1}/{U_{nn}} \end{bmatrix}$$

$\Rightarrow V = D^{-1}U$, Therefore, V becomes an upper triangular matrix.

This means $A = LDV$.

What needs to be done to define the proof is to show uniqueness. Let $A = L_1D_1V_1$ and $A = L_2D_2V_2$, let $U = D_1V_1$, $U = D_2V_2$. Both U_1 and U_2 are upper triangular matrices. L_1 and L_2 matrices are lower triangular matrices. From the previous theorem, the necessity and necessity of $L_1 = L_2$ and $U_1 = U_2$ emerge.

This means $D_1V_1 = D_2V_2$.

$$D_2^{-1}D_1 = V_2V_1^{-1} \Rightarrow D_2^{-1}D_1 = I = V_2V_1^{-1}$$

$$D_1 = D_2 \text{ and } V_1 = V_2$$

7.0 CROUT METHOD

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & \cdots & \cdots & 0 \\ l_{21} & l_{22} & 0 & \cdots & 0 \\ l_{31} & l_{32} & l_{33} & \ddots & \vdots \\ \vdots & \vdots & \ddots & \vdots & 0 \\ l_{n1} & l_{n2} & \cdots & \cdots & l_{nn} \end{bmatrix} \begin{bmatrix} 1 & U_{12} & \cdots & \cdots & U_{1n} \\ 0 & 1 & \cdots & \cdots & U_{2n} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & 1 \end{bmatrix}$$

$$A = LU$$

The first column of matrix U is completely known and has only one non-0 component. If we multiply all rows of matrix L by the first column of U; $a_{i1} = l_{i1}$; We obtain the equality $i = 1,2,..n$.

Therefore, the first column of L is determined. Now, let's multiply the first row of matrix L and the columns after the first of matrix U.

$$a_{1j} = l_{11}U_{1j}, \;\; j = 2,3, \dots, n$$

$U_{1j} = {a_{1j}}/{l_{11}}$, $j = 2,3, \dots, n$ the first line of U becomes clear.

If we continue in a similar way, 2 columns of L are determined, and then 2 rows of U are determined. If we generalize:

Let the first $(k-1)$ column of L and the first $(k-1)$ row of U be determined. Let's multiply all the rows of the L matrix starting from k with the k columns of the U matrix.

$$a_{ik} = \sum_{m=1}^{k} l_{im}U_{mk} \;\; , \;\; i = k, \dots, n$$

$$a_{ik} = \sum_{m=1}^{k-1} l_{im}U_{mk} + l_{ik}U_{kk} = \sum_{m=1}^{k-1} l_{im}U_{mk} + l_{ik}$$

The $l_{im}'s$ in the $\sum$ operation are from the first $(k-1)$ column of the L matrix. Therefore, their values are known. Umk is from the first $(k-1)$ row of the U matrix and therefore its values appear. Therefore, the k columns of the L matrix are determined. This time, if the k rows of the L matrix are multiplied by the k columns of the U matrix.

$$a_{kj} = \sum_{m=1}^{k} l_{km}U_{mj} \;\; , \;\; j = k+1, \dots, n$$

$$a_{kj} = \sum_{m=1}^{k-1} l_{km}U_{mj} + l_{kk}U_{kj} \, , \;\; j = k+1, \dots, n$$

The l_{mk}'s in the $\sum$ operation are from the first $(k-1)$ column of the L matrix and therefore their values are known. Umj's are from the first $(k-1)$ row of the U matrix and therefore their values are known. At the same time, the $l_{kk} \neq 0$ value was determined in the previous step.

$$U_{kj} = (a_{kj} - \sum_{m=1}^{k-1} l_{km}U_{mj})/l_{kk} \;\; ; \;\; j = k+1, \dots, n$$

and the k rows of the U matrix have been determined.

Example 7.1.

$$\begin{bmatrix} 6 & 2 & 1 & -1 \\ 2 & 4 & 1 & 0 \\ 1 & 1 & 4 & 4 \\ -1 & 0 & -1 & 3 \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 & 0 \\ l_{21} & l_{22} & 0 & 0 \\ l_{31} & l_{32} & l_{33} & 0 \\ l_{41} & l_{42} & l_{43} & l_{44} \end{bmatrix} \begin{bmatrix} 1 & U_{12} & U_{13} & U_{14} \\ 0 & 1 & U_{23} & U_{24} \\ 0 & 0 & 1 & U_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

First, let's multiply the rows of L by the first column of U.

$$l_{11} = 6 \, , l_{21} = 2 \, , l_{31} = 1 \, , l_{41} = -1$$

Let's multiply the first row of L by the other columns of U.

$$l_{11}U_{12} = 2\,, l_{11}U_{13} = 1\,, l_{11}U_{14} = -1$$

$$U_{12} = \frac{2}{l_{11}}\,, U_{13} = \frac{1}{l_{11}}\,, U_{14} = -\frac{1}{l_{11}}$$

$$U_{12} = \frac{2}{6} = \frac{1}{3}\,, U_{13} = \frac{1}{6}\,, U_{14} = -\frac{1}{6}$$

Let's multiply all the rows of L starting from 2 by the 2 columns of U.

$$l_{22} = a_{22} - l_{21}U_{12} \Rightarrow l_{22} = \frac{10}{3}$$

$$l_{32} = a_{32} - l_{31}U_{12} \Rightarrow l_{32} = \frac{2}{3}$$

$$l_{42} = a_{42} - l_{41}U_{12} \Rightarrow l_{42} = \frac{1}{3}$$

Now let's multiply the 2 rows of L by the 2 columns of U.

$$l_{21}U_{13} + l_{22}U_{23} = a_{23} \Rightarrow U_{23} = \frac{1}{5}$$

$$l_{21}U_{14} + l_{22}U_{24} = a_{24} \Rightarrow U_{24} = \frac{1}{10}$$

Continuing, if we multiply all the rows of L starting from 3 by the 3 columns of U.

$$l_{33} = a_{33} - (l_{31}U_{13} + l_{32}U_{23}) \Rightarrow l_{33} = \frac{37}{10}$$

$$l_{43} = a_{43} - (l_{41}U_{13} + l_{42}U_{23}) \Rightarrow l_{43} = -\frac{9}{10}$$

Now, if we multiply the 3 rows of L with the columns after 3 of U, that is, 4 columns, we get.

$$U_{32} = \frac{[a_{34} - (l_{31}U_{14} + l_{32}U_{24})]}{l_{33}} \Rightarrow U_{32} = -\frac{9}{37},$$

4 rows of L have l_{44} elements.

$$l_{44} = a_{44} - (l_{41}U_{14} + l_{42}U_{24} + l_{43}U_{34}) \Rightarrow l_{44} = \frac{191}{74}$$

$$\begin{bmatrix} 6 & 2 & 1 & -1 \\ 2 & 4 & 1 & 0 \\ 1 & 1 & 4 & 4 \\ -1 & 0 & -1 & 3 \end{bmatrix} = \begin{bmatrix} 6 & 0 & 0 & 0 \\ 2 & {}^{10}/_{3} & 0 & 0 \\ 1 & {}^{2}/_{3} & {}^{37}/_{10} & 0 \\ -1 & {}^{1}/_{3} & -{}^{9}/_{10} & {}^{191}/_{74} \end{bmatrix} \begin{bmatrix} 1 & {}^{1}/_{3} & {}^{1}/_{6} & -{}^{1}/_{6} \\ 0 & 1 & {}^{1}/_{5} & {}^{1}/_{10} \\ 0 & 0 & 1 & -{}^{9}/_{37} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Theorem 7.2.: Let A be a nxn symmetric matrix. If it can be factored as $A = LDL^T$, where L is a lower triangular matrix and D is a diagonal matrix, the necessary and sufficient conditions for A to be positive definite are that all diagonal components of D are positive.

Algorithm of subtraction of LDL^T

i) Dimension n is obtained.

ii) All components of matrix A are obtained.

iii) Repeat the following from $i = 1$ to n

1)$d_i = a_{ii} - \sum_{k=1}^{i-1} l_{ik}^2 d_k$

2) Repeat the following from $j = (i + 1)$ to n

$l_{ji} = (a_{ji} - \sum_{k=1}^{i-1} l_{jk} l_{ik} d_k)/d_i$

Example 7.3.

$$A = \begin{bmatrix} 4 & -1 & 1 \\ -1 & 4,25 & 2,75 \\ 1 & 2,75 & 3,50 \end{bmatrix}$$

$\underline{i = 1}$

$d_1 = a_{11} = 4$,

$l_{21} = {}^{a_{21}}/_{d_1} \Rightarrow l_{21} = -{}^{1}/_{4} = -0,25$,

$l_{31} = {}^{a_{31}}/_{d_1} \Rightarrow l_{21} = {}^{1}/_{4} = 0,25$

$\underline{i = 2}$

$d_2 = a_{22} - l_{21}^2 d_1 \Rightarrow d_2 = 4,25 - (-0,25)^2 \cdot 4 \Rightarrow d_2 = 4$

$l_{21} = {}^{a_{21}}/_{d_1} \Rightarrow l_{21} = -{}^{1}/_{4} = -0,25$

$$l_{32} = \frac{[a_{32} - l_{31} l_{21} d_1]}{d_2} \Rightarrow l_{32} = \frac{[2,75 - (0,25)(-0,25) \cdot 4]}{4} \Rightarrow l_{32} = \frac{3}{4} = 0,75$$

$\underline{i = 3}$

$d_3 = a_{33} - [l_{31}^2 d_1 + l_{32}^2 d_2] \Rightarrow d_3 = 3,5 - [(0,25)^2 \cdot 4 - (0,75)^2 \cdot 4] \Rightarrow d_3 = 1$

$$D = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad L = \begin{bmatrix} 1 & 0 & 0 \\ -0,25 & 1 & 0 \\ 0,25 & 0,75 & 1 \end{bmatrix}$$

Example 7.4.

$$A = \begin{bmatrix} 4 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

$\underline{i = 1}$

$d_1 = a_{11} = 4$,

$l_{21} = {}^{a_{21}}/_{d_1} \Rightarrow l_{21} = -{}^{1}/_{4} = -0{,}25$,

$l_{31} = {}^{a_{31}}/_{d_1} \Rightarrow l_{21} = 0$

$\underline{i = 2}$

$d_2 = a_{22} - l_{21}^2 d_1 \Rightarrow d_2 = 2 - (-0{,}25)^2 \cdot 4 \Rightarrow \; d_2 = 1{,}75$

$$l_{32} = \frac{[a_{32} - l_{31} l_{21} d_1]}{d_2} \Rightarrow l_{32} = \frac{[-1 - 0]}{1{,}75} \Rightarrow l_{32} = -\frac{4}{7} = -0{,}5714286$$

$\underline{i = 3}$

$$d_3 = a_{33} - [l_{31}^2 d_1 + l_{32}^2 d_2] \Rightarrow d_3 = 2 - \left[0 + \frac{16}{49} \cdot \frac{7}{4}\right] \Rightarrow d_3 = 2 - \frac{4}{7} \Rightarrow d_3 = 1{,}43$$

8.0 CHOLES METHOD

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} = \begin{bmatrix} g_{11} & 0 & \cdots & 0 \\ g_{21} & g_{22} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ g_{n1} & g_{n2} & \cdots & g_{nn} \end{bmatrix} \begin{bmatrix} g_{11} & g_{12} & \cdots & g_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & g_{nn} \end{bmatrix}$$

$$A = GG^T$$

A is symmetric and can be factored as $G = G^T$

1) Let's multiply all the rows of G by the first column of G^T.

$$a_{i1} = g_{i1} g_{11} \,, i = 1,2, \dots, n$$

$$\underline{i = 1} \quad a_{11} = g_{11}^2 \Rightarrow a_{11} = \sqrt{g_{11}}$$

$$i > 1 \quad g_{i1} = \frac{a_{i1}}{g_{11}}$$

2) Let's multiply all the rows of G starting from 2 by the second column of G^T.

$$a_{i2} = g_{i1} g_{21} + g_{i2} g_{22}$$

$$\underline{i = 2} \quad a_{22} = g_{21}^2 + g_{22}^2 \Rightarrow g_{22} = \sqrt{a_{22} - g_{21}^2}$$

$$i > 2 \quad g_{i2} = \frac{a_{i2} - g_{i1}g_{21}}{g_{22}}$$

3) Let the first $j-1$ column of G be determined. Let's multiply all the rows of G starting from j with the j column of G^T.

$$a_{ij} = \sum_{k=1}^{j} g_{ik}g_{jk} \quad i = j, j+1, \ldots, n$$

$$\underline{i = j}\, a_{jj} = \sum_{k=1}^{j} g_{jk}^2 \Rightarrow a_{jj} = \sum_{k=1}^{j-1} g_{jk}^2 + g_{jj}^2 \Rightarrow g_{jj} = \sqrt{a_{jj} - \sum_{k=1}^{j-1} g_{jk}^2}$$

$$i > j \quad g_{ij} = \frac{a_{ij} - \sum_{k=1}^{j-1} g_{ik}g_{jk}}{g_{jj}}$$

Example 8.1.

$$A = \begin{bmatrix} 4 & -1 & 1 \\ -1 & 4,25 & 2,75 \\ 1 & 2,75 & 3,50 \end{bmatrix} = GG^T$$

$$i = 1 \Rightarrow \quad a_{11} = g_{11}^2 \Rightarrow a_{11} = \sqrt{g_{11}} \quad \Rightarrow a_{11} = \sqrt{4} \Rightarrow a_{11} = 2$$

$$i > 1 \quad g_{i1} = \frac{a_{i1}}{g_{11}}$$

$$g_{21} = \frac{a_{21}}{g_{11}} \Rightarrow g_{21} = -\frac{1}{2} = -0,5\,, \qquad g_{31} = \frac{a_{31}}{g_{11}} \Rightarrow g_{31} = \frac{1}{2} = 0,5$$

$$i = 2\_ \quad a_{22} = g_{21}^2 + g_{22}^2 \Rightarrow g_{22} = \sqrt{a_{22} - g_{21}^2} \Rightarrow g_{22} = \sqrt{4,25 - (-0,5)^2} \Rightarrow g_{22} = \sqrt{4} = 2$$

$$i > 2 \quad g_{i2} = \frac{a_{i2} - g_{i1}g_{21}}{g_{22}}$$

$$g_{32} = \frac{a_{32} - g_{31}g_{21}}{g_{22}} \Rightarrow g_{32} = \frac{2,75 - 0,5 \cdot (-0,5)}{2} \Rightarrow g_{32} = \frac{3}{2} = 1,50$$

$$i = 3 \Rightarrow g_{33} = \sqrt{a_{33} - (g_{31}^2 + g_{32}^2)} \Rightarrow g_{33} = \sqrt{3,50 - (0,50^2 + 1,50^2)} \Rightarrow g_{33} = \sqrt{1} = 1$$

9.0 DISCUSSION AND CONCLUSIONS

The purpose of this paper is to show the details of matrix algebra, matrix factorization, definitions of upper and lower triangular matrices with several methods such as Doolittle method, Crout method, Choles method and examples of their application.

From the above, we tried to bring the examples closer to the smallest detail. To show examples that can be applied by doing these methods. It is hoped that this part of the chapter will continue and be useful to all those who want to know more about matrices and the method of solving them...

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